3501. This is false. A counterexample is $y = x+2$, which has a y intercept at $y = 2$. The graph $y^2 = x + 2$ has y intercepts at $y = \pm\sqrt{2}$.

3502. Both top and bottom are zero at $x = 0.2$. Hence, using the factor theorem, $(5x-1)$ must be a factor of both:

$$\begin{aligned}&\lim_{x\to 0.2} \frac{15x^3 + 2x^2 + 4x - 1}{5x^2 + 4x - 1}\\ &= \lim_{x\to 0.2} \frac{(5x-1)(3x^2 + x + 1)}{(5x-1)(x+1)}\\ &= \lim_{x\to 0.2} \frac{3x^2 + x + 1}{x+1}\\ &= \tfrac{11}{10}.\end{aligned}$$

3503. Assume that each region is equally likely to be shaded or unshaded. Then there are 16 outcomes in the possibility space. Classifying by number of squares shaded:

$$\begin{aligned}\le 1 \text{ shaded} &: 0 \text{ successful outcomes}\\ 2 \text{ shaded} &: 4 \text{ successful outcomes}\\ 3 \text{ shaded} &: 4 \text{ successful outcomes}\\ 4 \text{ shaded} &: 1 \text{ successful outcome.}\end{aligned}$$

So, the probability p is $\frac{9}{16}$.

3504. This is false. As a counterexample, consider

$$\begin{aligned}y &= (x-1)(x-2)(x-3),\\ y &= -(x-1)(x-2)(x-3).\end{aligned}$$

These are different cubics (their y intercepts are ∓ 6), and they intersect at three distinct points $(1, 0)$, $(2, 0)$ and $(3, 0)$.

3505. These are quadratics of the form $ax^2 + bxy + cy^2$. We can determine the possibility of factorisation by consider the discriminant $\Delta = b^2 - 4ac$:

(a) $\Delta = -4 < 0$, so not factorisable.
(b) $\Delta = -3 < 0$, so not factorisable.
(c) $\Delta = 0$, so factorisable as $(x + y)^2$.

3506. Differentiating implicitly,

$$\begin{aligned}&x - y + \sqrt{x+y} = 0\\ \Longrightarrow\ &1 - \tfrac{dy}{dx} + \tfrac{1}{2}(x+y)^{-\frac{1}{2}}\big(1 + \tfrac{dy}{dx}\big) = 0.\end{aligned}$$

Looking for tangents of the form $y = x + k$, we set $\frac{dy}{dx} = 1$, which gives $(x + y)^{-\frac{1}{2}} = 0$. A reciprocal cannot equal zero, so this has no roots. Hence, no line of the form $y = x + k$ is tangent to the curve.

3507. (a) Solving $T^3 + T = 12T^2 - 47T + 64$ gives $T = 4$.

(b) The first model gives $\dot{x} = 3t^2 + 1$ and $\ddot{x} = 6t$. Evaluating these at $t = 4$, we get 49 and 24. The second model gives $\dot{x} = 24t - 47$ and $\ddot{x} = 24$. At $t = 4$, these are also 49 and 24.

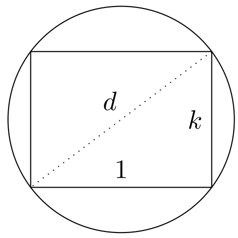
(c) At time $t = 4$, the positions are the same. At time $t = 14$, the predicted positions are 2758 and 1758. These predictions differ by 1000.

3508. A counterexample is

$$\begin{aligned}x + y + z &= 1,\\ x + y + z &= 2,\\ x + y + z &= 3.\end{aligned}$$

There are no (x, y, z) solution points common to all three. Graphically, these are the equations of three parallel planes in (x, y, z) space.

3509. (a) The scenario is



The diameter is $d = \sqrt{k^2 + 1}$. So, the radius is $r = \frac{1}{2}\sqrt{k^2 + 1}$. Hence, the area of the circle is $\frac{1}{4}\pi(k^2 + 1)$. The area of the rectangle is k. So, the ratio of areas is $k : \frac{1}{4}\pi(k^2 + 1)$.

(b) Dividing the ratio by k, it is

$$1 : \tfrac{1}{4}\pi\big(k + \tfrac{1}{k}\big).$$

So, we define z as the RHS of the above. Then the ratio of areas is optimised when $\frac{dz}{dk} = 0$. Setting this derivative to zero,

$$\begin{aligned}&\tfrac{1}{4}\pi\big(1 - \tfrac{1}{k^2}\big) = 0\\ \therefore\quad &k = 1.\end{aligned}$$

The second derivative is

$$\frac{d^2 z}{dk^2} = \tfrac{1}{2}\pi k^{-3}.$$

This is positive at $k = 1$. So, $k = 1$ minimises the ratio of areas, as required.

3510. Firstly, we rearrange to $\mathrm{f}(x) = -\ln(1 + x)$. Then, over the domain $(-1, \infty)$, the input $(1 + x)$ has range $(0, \infty)$, which is the standard domain of the natural logarithm. This has range $\mathbb{R}$. Hence, so does $-\ln(1+x)$. Furthermore, the function is one-to-one, so invertible.

To find f^{-1}, we set $y = -\ln(1+x)$, so $x = e^{-y} - 1$. Switching the domain and codomain, the formal definition of the inverse function is

$$\mathrm{f}^{-1} : \begin{cases}\mathbb{R} \longmapsto (-1, \infty),\\ x \longmapsto e^{-x} - 1.\end{cases}$$

3511. The LHS is an infinite geometric series, with first term $(1+x^2)$ and common ratio x. Quoting the result for S_∞,

$$\begin{aligned}\frac{1+x^2}{1-x} &= x+2\\ \Longrightarrow x &= -1, \tfrac{1}{2}.\end{aligned}$$

We reject the former root, as it gives a common ratio of -1, for which the sum doesn't converge. Hence, the solution is $x = \frac{1}{2}$.

3512. (a) We have $\mathbb{P}(A_1) = 1/2$ and $\mathbb{P}(A_2) = 1/4$. Since they are independent, $\mathbb{P}(A_1 \cap A_2) = 1/8$. Using $\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2)$, we get $\mathbb{P}(A_1 \cup A_2) = 5/8$.

(b) The probability that no A_i occurs is

$$\tfrac{1}{2} \cdot \tfrac{3}{4} \cdot \tfrac{7}{8} \cdot \tfrac{15}{16} = \tfrac{315}{1024}.$$

Subtracting this from 1, the probability that at least one occurs is $\frac{709}{1024}$.

3513. The formula is

$$\int \frac{\mathrm{g}'(x)}{\mathrm{g}(x)}\,dx = \ln|\mathrm{g}(x)| + c.$$

The justification is differentiation of the result by the chain rule, which gives

$$\frac{d}{dx}\ln|\mathrm{g}(x)| = \frac{1}{\mathrm{g}(x)} \cdot \mathrm{g}'(x).$$

3514. From $\mathrm{f}(0) = 0$, we know that the coefficient $B = 0$. Differentiating, $\mathrm{f}'(0) = Ak = 1$, so $A = \frac{1}{k}$. Hence, the curve in question is

$$y = \tfrac{1}{k}\sin kx.$$

This is a standard sine wave, enlarged in both x and y directions by scale factor $\frac{1}{k}$.

(a) The first x intercept of $y = \sin x$ is at $x = \pi$. Enlarging by $\frac{1}{k}$ gives $\left(\frac{\pi}{k}, 0\right)$.

(b) The first turning point of $y = \sin x$ is at $\left(\frac{\pi}{2}, 1\right)$. Enlarging by $\frac{1}{k}$ gives $\left(\frac{\pi}{2k}, \frac{1}{k}\right)$.

3515. The ratio between each pair of terms is r^2, where r is the common ratio. Equating these,

$$\begin{aligned}\frac{x}{2x-3} &= \frac{2x+3}{x}\\ \Longrightarrow x &= \pm\sqrt{3}.\end{aligned}$$

Using the fourth and sixth terms, this gives the squared common ratio as

$$\begin{aligned}r^2 &= \frac{\pm 2\sqrt{3}+3}{\pm\sqrt{3}}\\ &= 2 \pm \sqrt{3}.\end{aligned}$$

Both of these values for r^2 are positive, so there are four possible values for r, as required.

3516. Differentiating by the product rule,

$$\begin{aligned}&\mathrm{f}^{(k)}(x) = (x+k)e^x\\ \Longrightarrow\; &\mathrm{f}^{(k+1)}(x) = e^x + (x+k)e^x\\ &\qquad = (x+k+1)e^x, \text{ as required.}\end{aligned}$$

3517. If M is large enough, the system will accelerate to the right, with friction at F_{max} acting up the slope. The reaction is $Mg\cos\theta$, so $F_{\text{max}} = \mu Mg\cos\theta$. The equation of motion along the string is then

$$Mg\sin\theta - \mu Mg\cos\theta - mg = (M+m)a.$$

Dividing by the total mass $(M+m)$ gives

$$a = \frac{Mg\sin\theta - \mu Mg\cos\theta - mg}{M+m}, \text{ as required.}$$

3518. The apparent asymmetry lies in the fact that y is given as a function of x. Each transformation can be thought of symmetrically as a replacement:

- Replacing x by $x-2$ gives $y = \mathrm{f}(x-2)$, which has been translated by 2 units in the positive x direction.
- Replacing y by $y-2$ gives $y-2 = \mathrm{f}(x)$, which has been translated by 2 units in the positive y direction. Rearranging to $y = \mathrm{f}(x)+2$ is what introduces the apparent asymmetry.

3519. A regular hexagon is six equilateral triangles. Each of these has side length $\sqrt{7^2+3^2} = \sqrt{58}$. Using the sine area formula, an equilateral triangle of side length l has area

$$A_\triangle = \tfrac{1}{2}l^2\sin 60° = \tfrac{\sqrt{3}}{4}l^2.$$

So, the area of the hexagon is

$$A_{\text{hex}} = 6 \cdot \tfrac{\sqrt{3}}{4} \cdot 58 = 87\sqrt{3}, \text{ as required.}$$

3520. (a) If $x \le 0$, then so must one of $x-y$ or $x+y$ be. But a logarithm only takes positive inputs.

(b) Using a log rule, $\ln\left(x^2-y^2\right) = 2$, which gives $x^2 - y^2 = e^2$. Substituting for y^2,

$$\begin{aligned}x^2 - (k-x) &= e^2\\ \Longrightarrow x^2 + x - e - e^2 &= 0.\end{aligned}$$

This is a quadratic in x. The formula gives

$$\begin{aligned}x &= \frac{-1 \pm \sqrt{1+4e+4e^2}}{2}\\ &= \frac{-1 \pm (1+2e)}{2}\\ &= -1-e, e.\end{aligned}$$

We reject the negative root, as described in part (a). So, the solution is $x = e$, $y = 0$.

3521. (a) This is false. A counterexample is

$$\mathrm{f}, \mathrm{g} : \begin{cases} x \mapsto |x|, & x \in [-1, 0], \\ x \mapsto 0, & \text{elsewhere.} \end{cases}$$

The function fg is zero everywhere.

(b) This is true. Since the range of f is $[0, 1]$, the composition fg cannot produce outputs outside $[0, 1]$.

3522. To produce a monic parabola with a minimum at $(p, -q)$, we need to reflect $y = -x^2 + ax + b$ in the x axis. This gives $y = x^2 - ax - b$.

3523. For $k > 0$, the inequality $X^2 > kX$ has solution set $(-\infty, 0) \cup (k, \infty)$. Using the symmetry of the normal distribution,

$$\begin{aligned} & \mathbb{P}\big(X^2 > kX\big) \\ = \ & \mathbb{P}\big(X \in (-\infty, 0) \cup (k, \infty)\big) \\ = \ & \mathbb{P}\big(X \in (-\infty, 0)\big) + \mathbb{P}\big(X \in (k, \infty)\big) \\ = \ & \tfrac{1}{2} + \mathbb{P}\big(X > k\big), \text{ as required.} \end{aligned}$$

3524. Let the angle of projection be θ. Vertically, the time of flight is given by

$$\begin{aligned} 0 &= (u \sin\theta)t - \tfrac{1}{2}gt^2 \\ \therefore \quad t &= \frac{2u\sin\theta}{g}. \end{aligned}$$

Substituting into the horizontal, the range is

$$\begin{aligned} d &= \frac{2u^2 \sin\theta\cos\theta}{g} \\ &\equiv \frac{u^2 \sin 2\theta}{g}. \end{aligned}$$

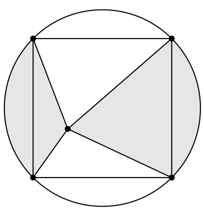
The maximum value of $\sin 2\theta$ is 1, so the maximum range is

$$d_{\max} = \frac{u^2}{g}, \text{ as required.}$$

3525. (a) This is true: square both sides.

(b) This is false. It holds if the second equation is well defined, but, for $a = b < 0$, neither $\sqrt{a}$ nor $\sqrt{b}$ is defined, so they cannot be equated.

3526. Connecting the points on the circumference to form a square, the scenario is



Let the square have side length 1, and the shaded triangles have widths x and $1-x$. Then the shaded triangles have combined area $\frac{1}{2}x + \frac{1}{2}(1-x) = \frac{1}{2}$. Hence, half of the square is shaded. Clearly half of the area outside the square is also shaded, which proves the result.

3527. Using the product rule,

$$\frac{dy}{dx} = (x+1)e^x.$$

At $x = -2$, $m_{\text{tan}} = -e^{-2}$. Hence, the normal has gradient $m_{\text{nor}} = e^2$. The equation of the normal through $\big(-2, -2e^{-2}\big)$ is therefore

$$\begin{aligned} & y + 2e^{-2} = e^2(x+2) \\ \Longrightarrow \ & y = e^2 x + 2e^2 - 2e^{-2} \\ \Longrightarrow \ & e^2 y = e^4 x + 2e^4 - 2, \text{ as required.} \end{aligned}$$

3528. Without loss of generality, place the vectors $\mathbf{a}$ and $\mathbf{b}$ on the axes, with $\mathbf{a} = a\mathbf{i}$ and $\mathbf{b} = b\mathbf{j}$. This gives

$$\begin{aligned} p\mathbf{a} + q\mathbf{b} &= pa\mathbf{i} + qb\mathbf{j}, \\ -q\mathbf{a} + p\mathbf{b} &= -qa\mathbf{i} + pb\mathbf{j}. \end{aligned}$$

These are perpendicular, so

$$\begin{aligned} & \frac{qb}{pa} \times \frac{pb}{-qa} = -1 \\ \Longrightarrow \ & \frac{b}{a} \times \frac{b}{a} = 1 \\ \Longrightarrow \ & b^2 = a^2. \end{aligned}$$

Hence, $|\mathbf{a}| = |\mathbf{b}|$, as required.

3529. (a) $\mathbb{P}(\text{RGB in that order}) = \frac{2}{6} \times \frac{2}{5} \times \frac{2}{4} = \frac{1}{15}$.

(b) $\mathbb{P}(\text{RGB in any order}) = 3! \times \frac{1}{15} = \frac{2}{5}$.

3530. The equation for intersections is

$$\begin{aligned} & x^2k^3 - 2x = 3k \\ \Longrightarrow \ & k^3x^2 - 2x - 3k = 0. \end{aligned}$$

If $k = 0$, then the equation is $-2x = 0$, giving one intersection at the origin. If $k \neq 0$, then the equation is a quadratic in x. The discriminant is $\Delta = 4 + 12k^4$. This is positive for any k, so there will always be two points of intersection. □

3531. Let $u = e^x$. Then $du = e^x\,dx$, so $dx = \frac{1}{u}\,du$. We can now enact the substitution:

$$\begin{aligned} & \int \frac{e^x + 1}{e^x + 4}\,dx \\ = & \int \frac{u+1}{u+4} \cdot \frac{1}{u}\,du \\ \equiv & \int \frac{u+1}{u(u+4)}\,du. \end{aligned}$$

In partial fractions, this is

$$\begin{aligned}&\int \frac{1}{4u}+\frac{3}{4(u+4)}\,du\\ &\equiv \tfrac{1}{4}\ln|u|+\tfrac{3}{4}\ln|u+4|+c\\ &= \tfrac{1}{4}x+\tfrac{3}{4}\ln(e^x+4)+c.\end{aligned}$$

———— Nota Bene ————

We can ditch the mod signs in the final expression, since e^x+4 is always positive.

3532. The intersections are at $x\ln x - x = 0$, which is $x(\ln x - 1) = 0$, so $x = 0, e$. Hence, the area is given by

$$A=\int_0^e x(1-\ln x)\,dx.$$

We integrate this by parts. Let $u = 1-\ln x$ and $v' = x$, so that $u' = -\frac{1}{x}$ and $v = \frac{1}{2}x^2$. This gives

$$\begin{aligned}A&=\left[\tfrac{1}{2}x^2(1-\ln x)\right]_0^e+\int_0^e \tfrac{1}{2}x\,dx\\ &=\left[\tfrac{3}{4}x^2-\tfrac{1}{2}x^2\ln x\right]_0^e\\ &=\tfrac{1}{4}e^2.\end{aligned}$$

3533. (a) For fixed points, $x = \mathrm{f}(x)$, which is

$$0=\frac{1}{x^2+1}.$$

But such a fraction cannot be zero, so f has no fixed points, as required.

(b) For fixed points, $x^4 - x^2 + 1 = 0$. This is a quadratic in x^2, with $\Delta = -3$. It has no real roots, so g has no fixed points, as required.

3534. We can ignore the exponential factor, as it is +ve everywhere. The boundary equation is

$$\begin{aligned}&x^2-2x-8=0\\ \Longrightarrow\ &x=-2,4.\end{aligned}$$

Hence, the solution is $x \in (-\infty,-2)\cup(4,\infty)$.

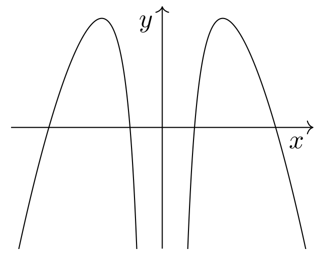
3535. (a) Writing $\tan x \equiv \frac{\sin x}{\cos x}$,

$$\begin{aligned}\frac{d}{dx}(\tan x)&\equiv\frac{\cos^2 x+\sin^2 x}{\cos^2 x}\\ &\equiv \sec^2 x,\ \text{as required.}\end{aligned}$$

(b) The integrand is $\frac{\mathrm{f}'(x)}{\mathrm{f}(x)}$. The standard result is

$$\int\frac{\sec^2 x}{\tan x}\,dx\equiv\ln|\tan x|+c.$$

3536. This is true if f has no discontinuities. So, we find a counterexample with an asymptote, such as



3537. (a) Let p be the probability that fungus is found in any square metre (in the population of square metres) of forest floor. The hypotheses are

$$\begin{aligned}H_0&: p=0.28,\\ H_1&: p\neq 0.28.\end{aligned}$$

(b) Assuming $X \sim B(40, 0.28)$, the expected value is $np = 8$. So, we calculate $\mathbb{P}(X \geq 19)$, which is 0.00677. For a two-tail test at the 1% level, we compare the p-value to 0.5%:

$$0.00677 > 0.005.$$

So, there is insufficient evidence to reject H_0: a sample of 19 out of 40 is not extreme enough to cast doubt on $B(40, 0.28)$. The biologist does not have sufficient evidence to claim a change in the prevalence of the fungus.

3538. Writing the binomial coefficients in factorials,

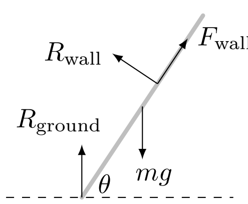
$$\begin{aligned}&{}^n\mathrm{C}_3+{}^n\mathrm{C}_4={}^n\mathrm{C}_5\\ \Longrightarrow\ &\frac{n!}{3!(n-3)!}+\frac{n!}{4!(n-4)!}=\frac{n!}{5!(n-5)!}\\ \Longrightarrow\ &\frac{1}{(n-3)(n-4)}+\frac{1}{4(n-4)}=\frac{1}{4\cdot 5}\\ \Longrightarrow\ &20+5(n-3)=(n-3)(n-4)\\ \Longrightarrow\ &n=6\pm\sqrt{29}.\end{aligned}$$

The binomial coefficients are defined for $n \in \mathbb{Z}^+$, so these roots do not satisfy the original equation: ${}^n\mathrm{C}_3 + {}^n\mathrm{C}_4 = {}^n\mathrm{C}_5$ has an empty solution set.

———— Nota Bene ————

We are justified in dividing by $n!$ in the above, because it cannot be zero.

3539. (a) The force diagram is



(b) Resolving perpendicular to the ladder,

$$\begin{aligned}&R_{\text{wall}}-mg\cos\theta+R_{\text{ground}}\cos\theta=0\\ \Longrightarrow\ &R_{\text{wall}}=\cos\theta(mg-R_{\text{ground}}).\end{aligned}$$

Resolving parallel to the ladder,

$$F_{\text{wall}} - mg\sin\theta + R_{\text{ground}}\sin\theta = 0$$
$$\implies F_{\text{wall}} = \sin\theta(mg - R_{\text{ground}}).$$

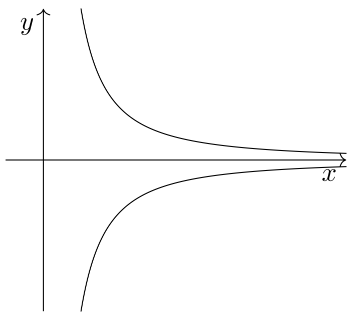
(c) Assuming that the ladder is on the point of slipping, $F_{\text{wall}} = \mu R_{\text{wall}}$. Hence,

$$\begin{aligned}\mu &= \frac{F_{\text{wall}}}{R_{\text{wall}}}\\ &= \frac{\sin\theta(mg - R_{\text{ground}})}{\cos\theta(mg - R_{\text{ground}})}\\ &\equiv \tan\theta.\end{aligned}$$

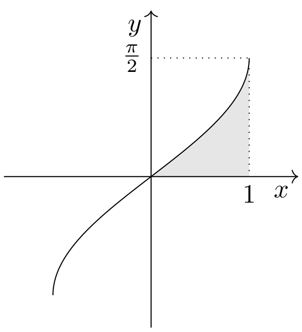
So, for equilibrium, we require $\mu \geq \tan\theta$.

3540. Solving the equation, $x^3 = 0, \pi, 2\pi, 3\pi$ and so on. Hence, the sequence x_n is given by $x_n = \sqrt[3]{n\pi}$. Since $n\pi$ can be made arbitrarily large, so can x_n. Therefore, the sequence diverges, as required.

3541. Making x the subject, we have $x = y^{-\frac{2}{3}}$. In the +ve quadrant, this is akin to the reciprocal graph $x = y^{-1}$. And, because the numerator of $\frac{2}{3}$ is even, the graph is symmetrical in the x axis: positive and negative values of y yield the same x value. So, the graph is



3542. (a) The graph of $y = \arcsin x$ is a reflection of (a restricted version of) $y = \sin x$ in the line $y = x$:



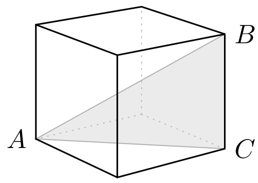
(b) I is given by the area of the rectangle formed by the dotted lines and the axes, minus the area between the curve and the y axis. This is

$$I = \tfrac{1}{2}\pi - \int_0^{\frac{\pi}{2}} \sin y\, dy.$$

(c) Evaluating the integral above,

$$I = \tfrac{\pi}{2} + \Big[\cos y\Big]_0^{\frac{\pi}{2}} = \tfrac{\pi}{2} - 1.$$

3543. Let the cube have unit side length. The relevant triangle consists of a space diagonal AB of length $\sqrt{3}$, an edge BC of length 1, and the remaining side is a face diagonal AC of length $\sqrt{2}$.



This is a right-angled triangle, as can be verified by Pythagoras. The angle between the edge and the space diagonal is given by

$$\arccos\frac{|AB|}{|BC|} = \arccos\tfrac{1}{\sqrt{3}}, \text{ as required.}$$

3544. (a) NII is $a = \sin 3t$. Integrating this,

$$v = -\tfrac{1}{3}\cos 3t + c.$$

Subbing $v_0 = 0$ gives $c = \frac{1}{3}$. Integrating again,

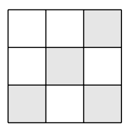
$$x = -\tfrac{1}{9}\sin 3t + \tfrac{1}{3}t + d.$$

Subbing $x_0 = 0$ gives $d = 0$. Hence,

$$x = -\tfrac{1}{9}\sin 3t + \tfrac{1}{3}t, \text{ as required.}$$

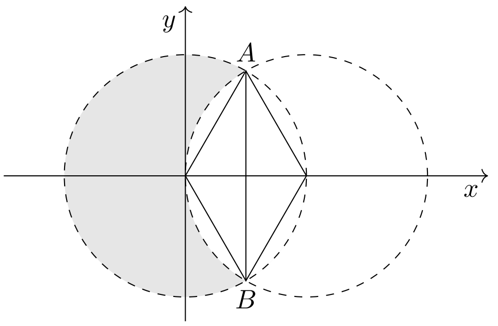
(b) The range of the velocity is $[-1/3, 1/3] + 1/3$, which is $[0, 2/3]$. So, the maximum speed in the motion is $2/3$ ms^{-1}.

3545. If the central square is shaded, then none of the centre edges can be. Three of the corners must be shaded, which gives four configurations of the following type:



If the central square is not shaded, then the rest of the squares must be alternately shaded/not shaded. There are two ways of doing this. Hence, there are six ways overall. QED.

3546. The boundary equations are circles of radius 6, centred at $(0, 0)$ and $(6, 0)$:



Chord AB subtends $\frac{2\pi}{3}$ at the centre of each circle. So, the area of a minor segment AB is

$$\begin{aligned}&\tfrac{1}{2}\cdot 6^2\left(\tfrac{2\pi}{3}-\sin\tfrac{2\pi}{3}\right)\\ &=12\pi-9\sqrt{3}.\end{aligned}$$

This gives the area of R as $36\pi - 2(12\pi - 9\sqrt{3})$, which simplifies to $12\pi + 18\sqrt{3}$.

3547. Writing $a \equiv e^{\ln a}$, we use the inverse chain rule:

$$\begin{aligned}&\int a^x\,dx\\ \equiv&\int e^{x\ln a}\,dx\\ \equiv&\frac{e^{x\ln a}}{\ln a}+c\\ \equiv&\frac{a^x}{\ln a}+c,\text{ as required.}\end{aligned}$$

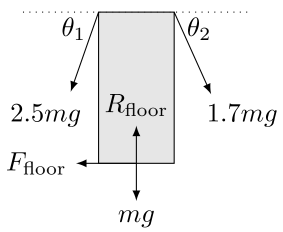
3548. For intersections, $x^3 - 3x^2 - 9x - 5 = 0$. Having used a polynomial solver to get $x = -1, 5$, we can factorise as $(x+1)^2(x-5) = 0$. This tells us that the line is tangent to the curve at $P : (-1, -4)$ and intersects it at $Q : (5, 50)$.

3549. Rearranging, we have a quadratic in $\sin\theta$:

$$\begin{aligned}&60\sin^2\theta + 29\sin\theta - 12 = 0\\ \Longrightarrow\ &(15\sin\theta - 4)(4\sin\theta - 3) = 0\\ \Longrightarrow\ &\sin\theta = \tfrac{4}{15}, \tfrac{3}{4}\\ \therefore\ &\theta = 15.5°, 48.6°, 131.4°, 164.5° \text{ (1dp)}.\end{aligned}$$

3550. (a) The tensions are different in the two sections. Therefore, the top of the load must be exerting a frictional force on the string. By NIII, the string must be exerting a frictional force on the load. Since the load is in equilibrium, the floor must also be exerting a horizontal force. So, none of the contacts are smooth.

(b) The forces on the load are



Vertically, $R_{\text{floor}} - mg - 3mg = 0$, so

$$R_{\text{floor}} = 4mg \text{ N}.$$

The horizontal components of tension are $\sqrt{2.5^2 - 1.5^2}mg$ and $\sqrt{1.7^2 - 1.5^2}mg$, which simplify to $2mg$ and $0.8mg$. The frictional force exerted by the floor is the difference:

$$F_{\text{floor}} = 1.2mg \text{ N}.$$

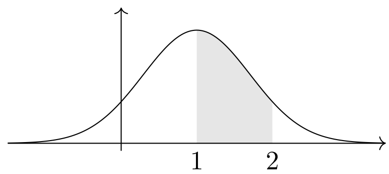
3551. The equation factorises $x(x^2y + y - 1) = 0$, which gives $x = 0$ or $x^2y + y - 1 = 0$. The former is a straight line, so contributes no points of inflection. The latter is

$$\begin{aligned}&y = \frac{1}{1+x^2}\\ \Longrightarrow\ &\frac{dy}{dx} = \frac{-2x}{(1+x^2)^2}\\ \Longrightarrow\ &\frac{d^2y}{dx^2} = \frac{6x^2-2}{(1+x^2)^3}.\end{aligned}$$

The second derivative is zero and changes sign at the two single roots of the numerator, where $x^2 = 1/3$. Substituting this in gives

$$y = \frac{1}{1+\frac{1}{3}} = \tfrac{3}{4}, \text{ as required.}$$

3552. Since the intervals $[0, 1]$ and $[1, 2]$ have the same width and probability, the mean must be $\mu = 1$:



Using an inverse normal, we need 2 to be 0.6745 standard deviations from the mean. So, we need $2 - 1 = 0.6745\sigma$, which gives $\sigma = 1.48$ (3sf).

3553. (a) Using the generalised binomial expansion,

$$\begin{aligned}&(9-24\lambda)^{\frac{1}{2}}\\ \equiv\ &3(1-\tfrac{8}{3}\lambda)^{\frac{1}{2}}\\ =\ &3(1-\tfrac{1}{2}\cdot\tfrac{8}{3}\lambda+...)\\ =\ &3-4\lambda+...\end{aligned}$$

For small λ, terms in λ^2 and above may be neglected, giving the required approximation.

(b) Solving for intersections,

$$\begin{aligned}&\lambda x^2 - 3x + 6 = 0\\ \Longrightarrow\ &x = \frac{3 \pm \sqrt{9-24\lambda}}{2\lambda}.\end{aligned}$$

If λ is close to zero, then $\sqrt{9-24\lambda} \approx 3 - 4\lambda$. Substituting this in,

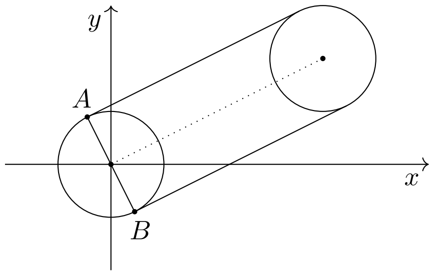
$$\begin{aligned}x &\approx \frac{3 \pm (3-4\lambda)}{2\lambda}\\ &\equiv 2, \tfrac{3}{\lambda} - 2.\end{aligned}$$

3554. This is a plan view, so the lines of action of weight and reaction forces are directly into the page.

(a) Taking moments around the x axis, we get $2R_1 = \frac{2}{3}W$, so $R_1 = \frac{1}{3}W$.

(b) Taking moments around the y axis, $3R_3 = W$, so $R_3 = \frac{1}{3}W$. Then, vertical equilibrium gives $R_1 + R_2 + R_3 = W$, so $R_2 = \frac{1}{3}W$.

3555. The scenario is



The dotted line of centres has gradient $\frac{1}{2}$. So, the diameter shown has gradient -2. Solving $y = -2x$ with $x^2 + y^2 = 1$ gives A and B as

$$\left(\pm\tfrac{1}{\sqrt{5}}, \mp\tfrac{2}{\sqrt{5}}\right).$$

So, the equations of the tangent lines are

$$y = \tfrac{1}{2}x \pm \tfrac{\sqrt{5}}{2}.$$

3556. Choosing the manager, then the deputies, then the secretaries, there are ${}^{25}\text{C}_1 \times {}^{24}\text{C}_2 \times {}^{22}\text{C}_3 = 10626000$ ways of selecting the six roles.

3557. Using the first-principles formula,

$$\begin{aligned}&\frac{d}{dx}\left(\frac{1}{\sqrt{x}+1}\right)\\ &\equiv \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}+1} - \frac{1}{\sqrt{x}+1}}{h}\\ &\equiv \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h(\sqrt{x}+1)(\sqrt{x+h}+1)}.\end{aligned}$$

We multiply top and bottom by the conjugate of the top, i.e. by $(\sqrt{x} + \sqrt{x+h})$. This gives

$$\begin{aligned}&\lim_{h\to 0} \frac{x - (x+h)}{h\big(\sqrt{x}+1\big)\big(\sqrt{x+h}+1\big)\big(\sqrt{x}+\sqrt{x+h}\big)}\\ \equiv\ &\lim_{h\to 0} \frac{-1}{\big(\sqrt{x}+1\big)\big(\sqrt{x+h}+1\big)\big(\sqrt{x}+\sqrt{x+h}\big)}.\end{aligned}$$

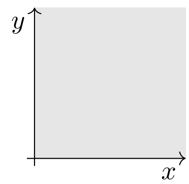
Having divided top and bottom by h, we take the limit. The first two brackets each tend to $(\sqrt{x}+1)$, and the third tends to $2\sqrt{x}$. This gives

$$\frac{d}{dx}\left(\frac{1}{\sqrt{x}+1}\right) = \frac{-1}{2(\sqrt{x}+1)^2\sqrt{x}}.$$

3558. Differentiating, $\text{f}'(x) = \cos x - \sqrt{3}\sin x$. This gives $\text{f}'(\pi) = -1$. Also $\text{f}(\pi) = -\sqrt{3}$. So, the best linear approximation g to f at $x = \pi$ is given by

$$\begin{aligned}&\text{g}(x) - \text{f}(\pi) = \text{f}'(\pi)(x - \pi)\\ \Longrightarrow\ &\text{g}(x) = -x + \pi - \sqrt{3}.\end{aligned}$$

3559. Using a log rule, $\ln {}^{x}\!/_{y} = k$, so ${}^{x}\!/_{y} = e^k$, which is $y = xe^{-k}$. We can express this as $y = Ax$, where $A \in (0, \infty)$. So, the loci are straight lines through the origin with positive gradient. However, since the equation is only defined for $x, y > 0$, the region is only the positive quadrant:



The axes are excluded from the region.

3560. The implication is forwards. If $\text{f}(x)$ has a factor of $(x-p)^2$, then $\text{f}(x) = 0$ has (at least) a double root at $x = p$, which means a stationary point. In the other direction, a counterexample is $\text{f}(x) = x^2 + 1$. This has a stationary point at $x = 0$, but doesn't have a factor of x^2.

3561. Writing the sum longhand,

$$\begin{aligned}&\frac{1}{2x+3} + \frac{1}{2x+1} + \frac{1}{2x-1} = 0\\ \Longrightarrow\ &(2x+1)(2x-1) + (2x+3)(2x-1)\\ &\qquad + (2x+3)(2x+1) = 0\\ \Longrightarrow\ &12x^2 + 12x - 1 = 0\\ \Longrightarrow\ &x = -\tfrac{1}{2} \pm \tfrac{\sqrt{3}}{3}.\end{aligned}$$

3562. The invertible section of the $y = \cot x$ graph has domain $(0, \pi)$. Differentiating,

$$\begin{aligned}&y = \cot x\\ \Longrightarrow\ &\frac{dy}{dx} = -\operatorname{cosec}^2 x\\ \Longrightarrow\ &\frac{d^2y}{dx^2} = 2\cot x \operatorname{cosec}^2 x.\end{aligned}$$

The second derivative is zero at $x = \frac{\pi}{2}$. And it changes sign, as the factor of $\cot x$ is not repeated in the second derivative. Hence, $y = \cot x$ has a point of inflection at $({}^{\pi}\!/_{2}, 0)$.

Reflecting this in $y = x$, $y = \operatorname{arccot} x$ has a point of inflection at $(0, {}^{\pi}\!/_{2})$.

3563. Integrating the equations with respect to t,

$$2x + y = t + c,$$
$$x - y = d.$$

Substituting the latter into the former firstly for y, and secondly for x, gives, for new constants k_1, k_2:

$$x = \tfrac{1}{3}t + k_1,$$
$$y = \tfrac{1}{3}t + k_2.$$

At parameter $t = 3$, the coordinates are $(5, 0)$. This gives $k_1 = 4$ and $k_2 = -1$. The equations are

$$x = \tfrac{1}{3}t + 4,$$
$$y = \tfrac{1}{3}t - 1.$$

3564. Multiplying by $x^3 - 1$, we require

$$(Ax + B)(x - 1) + C(x^2 + x + 1) \equiv 6x^2 + 5x + 1.$$

Collecting terms, the LHS is

$$(A + C)x^2 + (-A + B + C)x + (-B + C).$$

Equating coefficients,

$$x^2 : A + C = 6$$
$$x^1 : -A + B + C = 5$$
$$x^0 : -B + C = 1.$$

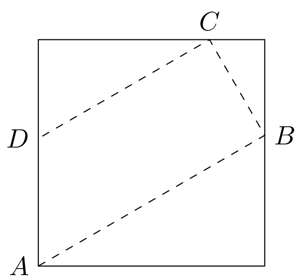
Adding all three equations, $3C = 12$, so $C = 4$, then $A = 2$ and $B = 3$.

3565. (a) P is not divisible by any p_i, because it is one more than a multiple of all of them. Hence, it has no prime factors other than itself, and is necessarily prime.

(b) P is larger than every element of the list, so it cannot be in the list.

(c) The list contains all the prime numbers, but P is another prime, which is not in the list. This is a contradiction. Hence, there are infinitely many prime numbers. QED.

3566. The nanobot's path is



Setting up the standard axes, we use the trig ratio $\tan 30° = \sqrt{3}/3$ to calculate coordinates as

$$B : \left(1, \tfrac{\sqrt{3}}{3}\right),$$
$$C : \left(\tfrac{4}{3} - \tfrac{\sqrt{3}}{3}, 1\right),$$
$$D : \left(0, \tfrac{4}{3} - \tfrac{4\sqrt{3}}{9}\right).$$

At D, the displacement is $s = \frac{4}{3} - \frac{4\sqrt{3}}{9}$ m.

3567. The initial speed is 28 ms^{-1} both horizontally and vertically. So, $x = 10 + 28t$ and $y = 28t - 4.9t^2$. The horizontal gives $t = \frac{x-10}{28}$. Substituting in,

$$\begin{aligned} y &= 28\left(\tfrac{x-10}{28}\right) - 4.9\left(\tfrac{x-10}{28}\right)^2 \\ &\equiv x - 10 - \tfrac{1}{160}x^2 + \tfrac{1}{8}x - \tfrac{5}{8} \\ &\equiv -\tfrac{1}{160}x^2 + \tfrac{9}{8}x - \tfrac{85}{8} \\ \Longrightarrow\ 160y &= -x^2 + 180x - 1700, \text{ as required.} \end{aligned}$$

3568. Substituting $-x$ for x,

$$\begin{aligned} y &= (-x)\left((-x)^2 - 1\right)^2 \\ &\equiv -x(x^2 - 1)^2. \end{aligned}$$

So, if (x, y) is a point on the curve, then $(-x, -y)$ is too. Hence, the curve has rotational symmetry about the origin, as required.

— Alternative Method —

When expanded, the curve is

$$y = x^5 - 2x^3 + x.$$

There are no terms of even degree in the expansion, so the curve has odd symmetry. This is rotational symmetry around the origin, as required.

3569. $\mathbb{P}$(SSSD in that order) is

$$1 \cdot \tfrac{12}{51} \cdot \tfrac{11}{50} \cdot \tfrac{39}{49} = 0.04120...$$

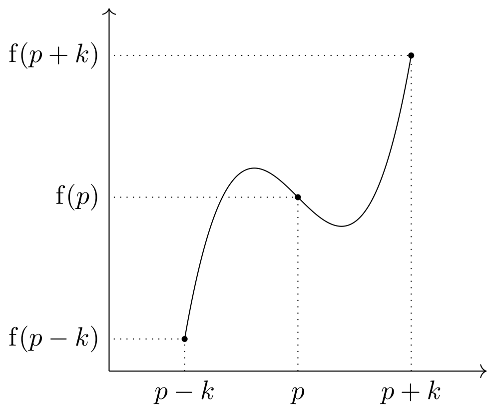
There are four possible orders of SSSD, so the total probability is $4 \cdot 0.0412 = 0.165$ (3sf).

— Alternative Method —

For a combinatorial approach, there are $^{52}\text{C}_4$ hands of cards. Of these, successful outcomes are SSSD. There are $4 \times {}^{13}\text{C}_3$ choices for S and then $3 \times {}^{13}\text{C}_1$ choices for D. So,

$$\mathbb{P}(\text{SSSD}) = \frac{4 \times {}^{13}\text{C}_3 \times 3 \times {}^{13}\text{C}_1}{^{52}\text{C}_4} = 0.165 \text{ (3sf)}.$$

3570. A cubic graph has rotational symmetry around its point of inflection. Hence, if we set p to be the x coordinate of the point of inflection of $y = \text{f}(x)$, then the x values $p \pm k$ are symmetrically either side of the centre of rotational symmetry.



Hence, $f(p+k)$ and $f(p-k)$ are equidistant from $f(p)$. So, their sum is $2\,f(p)$, which is independent of k, as required.

3571. ① This is true. Apart from $\sin x$, the other factor is $(\sin x - 2)$, which, since the range of $\sin x$ is $[-1, 1]$, cannot be zero.

② This is false. Apart from $\sin x$, the other factor is $(2\sin x - 1)$, which has roots. Any of these, such as $x = \pi/6$, provides a counterexample to the proposed implication.

3572. Factorising a cubic in $x^{\frac{1}{5}}$,

$$\begin{aligned} & x^{\frac{3}{5}} - 2x^{\frac{2}{5}} - 15x^{\frac{1}{5}} = 0 \\ \Longrightarrow\ & x^{\frac{1}{5}}(x^{\frac{2}{5}} - 2x^{\frac{1}{5}} - 15) = 0 \\ \Longrightarrow\ & x^{\frac{1}{5}}(x^{\frac{1}{5}} - 5)(x^{\frac{1}{5}} + 3) = 0 \\ \Longrightarrow\ & x^{\frac{1}{5}} = 0, 5, -3 \\ \Longrightarrow\ & x = 0, 3125, -243. \end{aligned}$$

3573. (a) Points on the unit circle may be parametrised by $x = \cos\theta$, $y = \sin\theta$. So, to calculate $\bar{Q}$, we find the continuous sum, for $\theta \in [0, 2\pi]$, of $x + y$, and then divide by the θ-width of the interval $[0, 2\pi]$. Writing x and y in terms of θ, this is

$$\bar{Q} = \frac{1}{2\pi}\int_0^{2\pi} \cos\theta + \sin\theta \, d\theta.$$

(b) Due to the symmetry of (the circle and thence) the cos and sin functions, the integral of each on $[0, 2\pi]$ is zero. Hence, $\bar{Q} = 0$.

3574. Differentiating by the chain rule,

$$\begin{aligned} & y = \sin^3 x + \cos 2x \\ \Longrightarrow\ & \frac{dy}{dx} = 3\sin^2 x \cos x - 2\sin 2x. \end{aligned}$$

Setting this to zero for SPs, and using a double-angle formula,

$$\begin{aligned} & \tfrac{3}{2}\sin x \sin 2x - 2\sin 2x = 0 \\ \Longrightarrow\ & \sin 2x(\tfrac{3}{2}\sin x - 2) = 0 \\ \Longrightarrow\ & \sin 2x = 0 \text{ or } \sin x = \tfrac{4}{3}. \end{aligned}$$

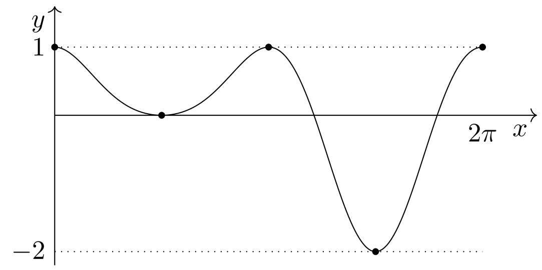
The latter equation has no roots, as the range of sine is $[-1, 1]$, which leaves $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$. The second derivative is

$$\frac{d^2y}{dx^2} = 6\sin x\cos^2 x - 3\sin^3 x - 4\cos 2x.$$

Evaluating at the SPs, we get $-4, 1, -4, 7, -4$. So, calculating the y coordinates, we have

local maxima at $(0, 1), (\pi, 1), (2\pi, 1)$,
local minima at $(\pi/2, 0), (3\pi/2, -2)$.

For $x \in [0, 2\pi]$, the graph is



3575. Denote the motions A for anticlockwise, B for no motion and C for clockwise. Successful outcomes are BBB, or ABC in any order. So, the probability p that the shading ends up where it began is

$$p = \tfrac{1}{2}^3 + 3! \cdot \tfrac{1}{4}^2 \cdot \tfrac{1}{2} = \tfrac{5}{16}.$$

3576. (a) The vertex is at $(1/2, \sqrt{3}/2)$. The line $y = \sqrt{3}x$ goes through the origin and this vertex, so it is one of the sides.

(b) The angle at O is $60°$, so the angle bisector is inclined at $30°$. The gradient is $\tan 30° = \sqrt{3}/3$, which gives the equation of the angle bisector as $y = (\sqrt{3}/3)x$.

(c) The angle bisector at $(1/2, \sqrt{3}/2)$ has equation $x = 1/2$. Solving this simultaneously with the equation of the angle bisector gives the height of the centre of the triangle as $y = \sqrt{3}/6$, which is $\frac{1}{3}$ of the height $\sqrt{3}/2$ of the triangle. Hence, the centre of the triangle divides its height in the ratio $1 : 2$.

3577. The product is 12, so the values showing may be $(1, 2, 6)$, $(1, 3, 4)$ or $(2, 2, 3)$ in any order. This gives $3! + 3! + 3 = 15$ outcomes in the possibility space. Of these, only the three orders of $(2, 2, 3)$ add to 7 and are successful. Hence, the probability is $3/15$, which is $1/5$.

3578. (a) Using the quotient rule, the first derivative is

$$\frac{dy}{dx} = \frac{x(x-2)e^x}{(x^2 + e^x)^2}.$$

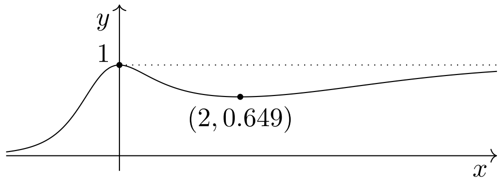
The numerator is zero at $x = 0$ and $x = 2$, so these are stationary points.

(b) We can take the limit $x \to -\infty$ immediately, giving $y \to 0$. To take the limit $x \to \infty$, we first divide top and bottom by e^x, giving

$$y = \frac{1}{x^2 e^{-x} + 1}.$$

Taking the limit as $x \to \infty$, we get $y \to 1$.

(c) There is a maximum at $(0, 1)$ and a minimum at $(2, 0.649)$. With the horizontal asymptotes from part (b), this gives



3579. (a) The angles add to 180°, so $C = 180° - A - B$. Using the symmetries of the tan function,

$$\begin{aligned} \tan C &= \tan(180° - A - B) \\ &\equiv \tan(-A - B) \\ &\equiv -\tan(A + B) \\ &\equiv -\frac{\tan A + \tan B}{1 - \tan A \tan B} \\ &\equiv \frac{\tan A + \tan B}{\tan A \tan B - 1}. \end{aligned}$$

(b) Using the result above,

$$\begin{aligned} &\tan A + \tan B + \tan C \\ = \;&\tan A + \tan B + \frac{\tan A + \tan B}{\tan A \tan B - 1} \\ \equiv \;&\frac{(\tan A + \tan B)(\tan A \tan B - 1) + \tan A + \tan B}{\tan A \tan B - 1} \\ \equiv \;&\tan A \tan B \frac{\tan A + \tan B}{\tan A \tan B - 1} \\ = \;&\tan A \tan B \tan C, \text{ as required.} \end{aligned}$$

3580. (a) i. $\mathbb{P}(M > 70) = 0.202$ (3sf).

ii. Assuming (ha!) that members of the group are independently chosen, the mean of a group of five will be $\bar{M} \sim N(60, {}^{12^2}/_5)$. So,

$$\mathbb{P}(\bar{M} > 70) = 0.0312 \text{ (3sf)}.$$

(b) There is no reason why a biological population should follow a normal distribution.

Furthermore, there will not be independence among the individual members of a group.

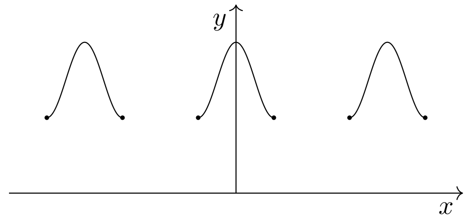
— Nota Bene —

In general, the use of the normal distribution is a mathematical convenience, and very rarely produces results which are practically useful. Numbers are cracking toys, but life is bigger.

3581. Squaring both sides gives $y - 1 = \cos^2 x$, which we rearrange to $y = \cos^2 x + 1$. Using a double-angle formula, this is a sinusoidal graph:

$$y = \tfrac{1}{2}(\cos 2x + 3).$$

However, by squaring we have introduced solution points. Where $\cos x$ is negative, there are no points satisfying $\sqrt{y - 1} = \cos x$. So, the graph is half of a sinusoidal graph:



3582. For the cubic to be invertible over $\mathbb{R}$, it must be one-to-one. Hence, it must have no turning points. This means it can have a maximum of one SP.

For SPs, $3x^2 + 2kx + 3 = 0$. We need this equation to have at most one root, so $\Delta = 4k^2 - 36 \leq 0$. This gives $k \in [-3, 3]$.

3583. (a) The derivative is $f'(x) = 3x^2 - 2$. This gives tangent lines $y = -2x + 2$ and $y = x$, with x intercepts $x = 1$ and $x = 0$ respectively.

(b) The N-R iteration finds the x intercept of the tangent line. Hence, from either $x_0 = 0$ or $x_0 = 1$, the sequence proceeds periodically:

$$x_n = ..., 0, 1, 0, 1, 0, ...$$

So, it cannot converge to a root.

3584. The interior angles of an n-gon add to $(n - 2)\pi$ radians. So, their mean is

$$\bar{\theta} = \frac{n - 2}{n}\pi < \pi.$$

In a regular n-gon, this mean is the largest angle. This is the lower bound (attainable) on the largest angle.

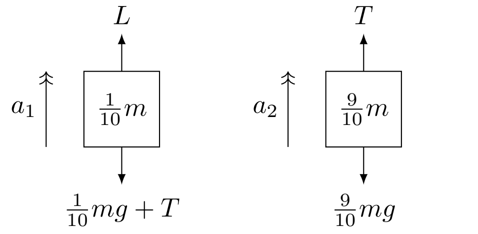
For the upper bound, $\bar{\theta}$ is closer to 0 than to 2π. Hence, the value of the upper bound is dictated by the fact that the smallest angle must be greater than 0. In the boundary case (not attainable), the smallest angle is zero. The largest angle is then the same amount above $\bar{\theta}$ as 0 is below it:

$$\theta = 2 \times \frac{n - 2}{n}\pi.$$

So, the largest angle must satisfy

$$\theta \in \left[\frac{n - 2}{n}\pi, \frac{2(n - 2)}{n}\pi\right).$$

3585. (a) Before the monkey starts to climb, the forces on the combined monkey-balloon are lift L and weight $\frac{1}{10}mg + \frac{9}{10}mg$. Hence, the lift is mg. When the monkey climbs, the forces are



For the balloon, $mg - \frac{1}{10}mg - \frac{6}{5}mg = \frac{1}{10}ma_1$, which gives $a_1 = -3$ ms^{-2}. For the monkey, $T - \frac{9}{10}mg = \frac{9}{10}ma_2$, which gives $a_2 = \frac{1}{3}$ ms^{-2}.

(b) The acceleration of the monkey relative to the balloon is $3 + \frac{1}{3} = \frac{10}{3}$ ms^{-2}. So, $2.4 = \frac{1}{2} \cdot \frac{10}{3}t^2$, which gives $t = 1.2$ seconds.

3586. (a) True. According to Pythagoras, this consists of all (x, y, z) points a distance 1 from O.

(b) False. It is, in fact, an octahedron. This can be seen from considering the equation in the first octant, which is $x + y + z = 1$. This is a plane, whose normal is the line $x = y = z$. In the positive octant (i.e. $x, y, z \geq 0$), this gives an equilateral triangle. By symmetry of the modulus function, the same appears in each of the eight octants of (x, y, z) space.

3587. To eliminate the negative powers, we multiply top and bottom by e^x, before factorising:

$$\begin{aligned} &\frac{e^{4x+1} - 9e^{-1}}{e^{2x} - 3e^{-1}} \\ \equiv\ &\frac{e^{4x+2} - 9}{e^{2x+1} - 3} \\ \equiv\ &\frac{(e^{2x+1} + 3)(e^{2x+1} - 3)}{e^{2x+1} - 3} \\ \equiv\ &e^{2x+1} + 3. \end{aligned}$$

3588. Let $y = 2x + 3$, so $dy = 2\,dx$. The limits are $y = 1$ to $y = 3$. This gives

$$\int_{-1}^{0} 2\ln(2x + 3)\,dx = \int_{1}^{3} \ln y\,dy.$$

We now integrate by parts. Let $u' = \ln y$ and $v' = 1$, so that $u' = {}^1\!/_y$ and $v = y$. The parts formula gives the integral as

$$\begin{aligned} &\Big[y \ln y\Big]_1^3 - \int_1^3 1\,dy \\ =\ &\Big[y \ln y - y\Big]_1^3 \\ =\ &(3\ln 3 - 3) - (\ln 1 - 1) \\ =\ &\ln 27 - 2, \text{ as required.} \end{aligned}$$

3589. (a) Statement S is false here. A counterexample is $\mathrm{f}(x) = e^x$, which is increasing everywhere, but whose range is $(0, \infty)$.

(b) Statement S is true here. A polynomial which is increasing for all x can have no SPs, so must have odd degree. Hence, its range is $\mathbb{R}$.

3590. The volume of a cone is $V = \frac{1}{3}\pi r^2 h$, so the total volume of water is

$$V = \frac{1}{3}\pi r^2(2r) = \tfrac{2}{3}\pi r^3.$$

When the depth of water in the cone is H, the height has been scaled by ${}^H\!/_{2r}$, so the volume has been scaled by ${}^{H^3}\!/_{8r^3}$. Hence, at this point, the volume in the spherical cap is

$$\begin{aligned} &\tfrac{2}{3}\pi r^3 - \tfrac{H^3}{8r^3} \cdot \tfrac{2}{3}\pi r^3 \\ \equiv\ &\tfrac{2}{3}\pi r^3 - \tfrac{1}{12}\pi H^3. \end{aligned}$$

With the spherical cap formula,

$$\begin{aligned} &\tfrac{2}{3}\pi r^2 h = \tfrac{2}{3}\pi r^3 - \tfrac{1}{12}\pi H^3 \\ \Longrightarrow\ &8r^2 h = 8r^3 - H^3 \\ \Longrightarrow\ &H^3 = 8r^2(r - h), \text{ as required.} \end{aligned}$$

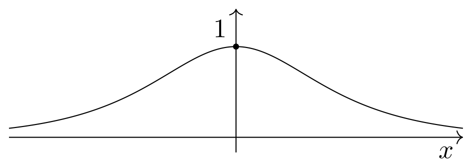
3591. Writing $\tan 3\theta$ as $\tan(2\theta + \theta)$, we expand with a compound-angle formula:

$$\begin{aligned} &\tan(2\theta + \theta) \\ \equiv\ &\frac{\tan 2\theta + \tan\theta}{1 - \tan 2\theta \tan\theta}. \end{aligned}$$

We then use a double-angle formula, giving

$$\begin{aligned} &\frac{\tan 2\theta + \tan\theta}{1 - \tan 2\theta \tan\theta} \\ \equiv\ &\frac{\frac{2\tan\theta}{1 - \tan^2\theta} + \tan\theta}{1 - \frac{2\tan\theta}{1 - \tan^2\theta}\tan\theta} \\ \equiv\ &\frac{2\tan\theta + \tan\theta(1 - \tan^2\theta)}{(1 - \tan^2\theta) - 2\tan^2\theta} \\ \equiv\ &\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}, \text{ as required.} \end{aligned}$$

3592. The denominator is symmetrical in $x = 0$. It has one turning point at $x = 0$, which is a minimum. The range of the denominator is $[2, \infty)$. The range of the full fraction, therefore, is $(0, 1]$. It has a local maximum at $(0, 1)$, does not cross the x axis, and $y \to 0$ as $x \to \pm\infty$. Putting these facts together, the graph is



— Nota Bene —

The RHS of the graph in this question has a name. It is a reciprocal hyperbolic trig function: sech x.

3593. This is true. The proposed solution $y = A\,\mathrm{f}(x)$ has derivative $\frac{dy}{dx} = A\,\mathrm{f}'(x)$. Substituting in,

$$\begin{aligned} &A\,\mathrm{f}'(x) + 3A\,\mathrm{f}(x) = 0 \\ \Longrightarrow\ &A\big(\mathrm{f}'(x) + 3\,\mathrm{f}(x)\big) = 0. \end{aligned}$$

Since $y = \mathrm{f}(x)$ satisfies the DE, the bracket is zero. Hence, regardless of the value of A, $y = A\,\mathrm{f}(x)$ is a solution curve.

3594. (a) Reflecting $y = \mathrm{f}(x)$ in $y = x$ gives $x = \mathrm{f}(y)$. Then reflecting in the y axis gives $x = -\mathrm{f}(y)$. The combined effect of these transformations is rotation by 90° anticlockwise around O.

(b) Since a positive odd polynomial connects $(-\infty, -\infty)$ with (∞, ∞), it must intersect with its rotation by 90°. Translations don't affect this, so the answers are

i. Yes,
ii. Yes,
iii. Yes.

3595. (a) Using the first Pythagorean identity,

$$\begin{aligned} &a \sin x = 1 - \cos^2 x \\ \Longrightarrow\ &a \sin x = \sin^2 x \\ \Longrightarrow\ &\sin x(\sin x - a) = 0 \\ \Longrightarrow\ &\sin x = 0, a. \end{aligned}$$

The equation $\sin x = 0$ has roots $x = n\pi$. But we are told that $x \neq n\pi$. Hence, $\sin x = a$.

(b) With $x \equiv 2 \cdot \frac{1}{2}x$, we use a double-angle formula and then divide by $\cos^2 \frac{1}{2}x$:

$$\begin{aligned} &2 \sin \tfrac{1}{2}x \cos \tfrac{1}{2}x = a \\ \Longrightarrow\ &2 \tan \tfrac{1}{2}x = a \sec^2 \tfrac{1}{2}x. \end{aligned}$$

(c) Using the second Pythagorean identity,

$$\begin{aligned} &2 \tan \tfrac{1}{2}x = a(1 + \tan^2 \tfrac{1}{2}x) \\ \Longrightarrow\ &a \tan^2 \tfrac{1}{2}x - 2 \tan \tfrac{1}{2}x + a = 0. \end{aligned}$$

The quadratic formula gives

$$\begin{aligned} \tan \tfrac{1}{2}x &= \frac{2 \pm \sqrt{4 - 4a^2}}{2a} \\ &\equiv \frac{1 \pm \sqrt{1 - a^2}}{a}, \text{ as required.} \end{aligned}$$

3596. Multiplying out the right-hand sides verifies the results: all terms by the first and last cancel.

(a) True,
(b) True,
(c) True.

— Nota Bene —

This is a general result:

$$x^n - 1 \equiv (x - 1)(x^{n-1} + x^{n-2} + ... + 1).$$

3597. Integrating by parts, the displacement of the first particle over $[0, t]$ is

$$s_1 = \int_0^t Te^T\, dT = (t - 1)e^t + 1.$$

The displacement of the second particle is

$$s_2 = \int_0^t (T - 1)e^T\, dT = (t - 2)e^t + 2.$$

There are now two possibilities, depending which way around the particles are at $t = 0$. The initial difference in position could be ± 1. So, we solve

$$\begin{aligned} &(t - 1)e^t + 1 = (t - 2)e^t + 2 \pm 1 \\ \Longrightarrow\ &e^t = 0, 2. \end{aligned}$$

Hence, if the particles start one way around, then they meet at $t = \ln 2$. But, if they start the other way around, then they never meet.

3598. This is false. A counterexample is $\mathrm{f}(x) = x + 1$. Such a linear function has odd degree (1), but has no fixed points, since $x + 1 \neq x$.

3599. (a) The derivative is $\frac{dy}{dx} = -5e^{-5x}$. This is equal to $-5y$, so the DE is satisfied.

(b) Differentiating by the product rule,

$$\begin{aligned} &y = \mathrm{g}(x)e^{-5x} \\ \Longrightarrow\ &\frac{dy}{dx} = \mathrm{g}'(x)e^{-5x} - 5\,\mathrm{g}(x)e^{-5x}. \end{aligned}$$

Substituting into the DE, we require

$$\begin{aligned} &\mathrm{g}'(x)e^{-5x} - 5\,\mathrm{g}(x)e^{-5x} \equiv -5\,\mathrm{g}(x)e^{-5x} \\ \Longrightarrow\ &\mathrm{g}'(x)e^{-5x} \equiv 0. \end{aligned}$$

Since e^{-5x} cannot be zero, this gives $\mathrm{g}'(x) \equiv 0$. Integrating, g is constant, as required.

3600. (a) Yes. In harmonic form, this is

$$y = \sqrt{2} \sin\big(x + \tfrac{\pi}{4}\big).$$

This has $x = \frac{\pi}{4}$ (and infinitely many others) as a line of symmetry.

(b) Yes. The y value is constant:

$$\arcsin x + \arccos x \equiv \tfrac{\pi}{2}.$$

So, the y axis and the line $y = \frac{\pi}{2}$ are lines of symmetry.

— End of 36th Hundred —